Elementary Submodels, the Löwenheim-Skolem-Tarski Theorems, and Models of Set Theory

Last week, we saw

- a construction of a formal proof theory and translations from informal mathematical reasoning to formal proofs.

- Discussed the Completeness theorem connecting model theory to formal proof theory. ($\mathrm{Con}_{\models}(\Sigma)$ iff $\mathrm{Con}_{\vdash,\mathcal{L}}(\Sigma)$ and $\Sigma \models \varphi$ iff $\Sigma \vdash_{\mathcal{L}} \varphi$.)

- Introduced some model-theoretic ideas. (elementary equivalence, isomorphic models, κ-categoricity of sets of sentences Σ of $\mathcal{L}$, completeness of Σ, and the Łoś-Vaught test for Completeness.

We'll continue with this last point, contuing to build our model theory foundations.

Extensions by definitions

Recall that Gabriel gave us a lexicon

$\mathcal{L} = \{\cdot, i, e\}$ and axioms

$GP = \{\gamma_1, \gamma_{2,1}, \gamma_{2,2}\}$ with

$$\gamma_1.\ \forall xyz[\ x\cdot(y\cdot z) = (x\cdot y)\cdot z],$$

$$\gamma_{2,1}.\ \forall x[x\cdot e = e\cdot x = x],$$

$$\gamma_{2,2}.\ \forall x[x\cdot i(x) = i(x)\cdot x = e].$$

These are axioms for groups.

Consider instead:

$\mathcal{L} = \{\cdot\}$ and

$GP' = \{\gamma_1, \gamma_2\}$, where

$$\gamma_1 = \forall xyz[x\cdot(y\cdot z) = (x\cdot y)\cdot z],$$

and

$$\gamma_2 = \exists u[\forall x[x\cdot u = u\cdot x = x] \wedge \forall x \exists y[x\cdot y = y\cdot x = u]]$$

We want these two to be equivalent!

The former just has some extra helpful symbols defined in $\mathcal{L}$ for identity and inverses.

Consider the case of ZFC, the development of which requires thousands of definitions added to $\in, =$.

This motivates our next definition:

Def$^{\underline{n}}$: Assume that $\mathcal{L} \subseteq \mathcal{L}'$ and Σ is a set of sentences (recall: a sentence is a Boolean-valued well-formed formula with no free variables) of $\mathcal{L}$.

If $p \in \mathcal{L}' - \mathcal{L}$ is an n-ary predicate symbol, then a <u>definition of p over $\mathcal{L}, \Sigma$</u> is a sentence of the form

$$\forall x_1, \ldots, x_n [p(x_1, \ldots, x_n) \longleftrightarrow \theta(x_1, \ldots, x_n)],$$

where θ is a formula of $\mathcal{L}$.

If $f \in \mathcal{L}' - \mathcal{L}$ is an n-ary function symbol, then a <u>definition of f over $\mathcal{L}, \Sigma$</u> is a sentence of the form

$$\forall x_1, \ldots, x_n [\theta(x_1, \ldots, x_n, f(x_1, \ldots, x_n))],$$

where θ is a formula of $\mathcal{L}$, and

$$\Sigma \vdash \forall x_1, \ldots, x_n \exists ! y \, \theta(x_1, \ldots, x_n, y).$$

A set of sentences Σ' of $\mathcal{L}$ is an <u>extension by definitions</u> of Σ if and only if $\Sigma' = \Sigma \cup \Delta$ where $\Delta = \{\delta_s : s \in \mathcal{L}'\}$ and each δ_s is a definition of s over $\mathcal{L}, \Sigma$.

Let's return to our group axioms example.

$GP' = \{\gamma_1, \gamma_2\}$ proves that the identity and inverses are unique.

Let

- $\theta_e(y)$ be $\forall x [x \cdot y = y \cdot x = x]$,
- $\theta_i(x,y)$ be $y \cdot (x \cdot x) = x$.

Then $GP = \{\gamma_1, \gamma_{2,1}, \gamma_{2,2}\}$ is just GP' with the axioms $\theta_e(e)$ and $\forall x \theta_i(x, i(x))$.

This doesn't really add more information, and that fact motivates our next theorem:

Theorem: Assume that $\mathcal{L} \subseteq \mathcal{L}'$, Σ is a set of sentences of $\mathcal{L}$, and Σ' in $\mathcal{L}'$ is an extension by definitions of Σ.

Not) Let $\forall\forall\chi$ denote some universal closure of a formula χ. (Recall that if χ

is a formula, a universal closure of χ is any sentence of the form

$$\forall x_1 \forall x_2 \ldots \forall x_n \varphi, \text{ where } n \geq 0.$$

Also recall that if χ is a formula, and sentences ψ, σ are universal closures of χ, then ψ and σ are logically equivalent.)

Then:

① If φ is any sentence of $\mathcal{L}$, then $\Sigma \vdash \varphi$ if and only if $\Sigma' \vdash \varphi$.

② If φ is any formula of $\mathcal{L}'$, then there is a formula $\hat{\varphi}$ of $\mathcal{L}$ with the same free variables such that

$$\Sigma' \vdash \forall\forall[\hat{\varphi} \leftrightarrow \varphi].$$

This means that $\hat{\varphi}$ and φ are equivalent with respect to Σ'. Recall from chapter II.8 of Kunen that equivalence means that if $\varphi, \hat{\varphi}$ are formulas of $\mathcal{L}'$ and Σ' is a set of sentences of $\mathcal{L}'$, then φ and $\hat{\varphi}$ are equivalent with respect to Σ' if and only if the universal closure of $\hat{\varphi} \longleftrightarrow \varphi$ is true in all models of Σ'. This is exactly what (2) says.

(3) If τ is any term of $\mathcal{L}'$, then there is a formula $\zeta_\tau(y)$ of $\mathcal{L}$ using the same variables as τ plus a new variable y such that

$$\Sigma \vdash \forall\forall \exists! y \, \zeta_\tau(y) \quad \text{and}$$
$$\Sigma' \vdash \forall\forall \, \zeta_\tau(\tau).$$

Overall, this means anything expressible with $\mathcal{L}'$ is also expressible with $\mathcal{L}$.

In fact, we can say the following, which is also important in the development of ZFC.

Lemma: Assume that $\mathcal{L} \subseteq \mathcal{L}' \subseteq \mathcal{L}''$, and that
- Σ is a set of sentences in $\mathcal{L}$,
- Σ' is a set of sentences in $\mathcal{L}'$,
- Σ'' is a set of sentences in $\mathcal{L}''$

such that Σ' is an extension by definitions of Σ, and Σ'' is an extension by definitions of Σ'. Then Σ'' is equivalent to an extension by definitions of Σ.

So chains of definitional extensions can be done with one step, as we did extending GP' to GP.

Elementary Submodels & Extensions

Neither of the previous talks defined "substructure" or "extension", so we'll do that now.

First, recall the definition of a structure for a lexicon $\mathcal{L}$.

Def$^{\underline{n}}$: Given a lexicon $\mathcal{L} = \mathcal{F} \cup \mathcal{P}$, a structure $\mathfrak{A}$ for $\mathcal{L}$ is a pair

$\mathfrak{A} = (A, I)$ such that A is a nonempty set and I is a function with domain $\mathcal{L}$ and each $I(s)$ is a semantic entity such that:

- if $f \in \mathcal{F}_n, n>0$, $I(f): A^n \longrightarrow A$.
- if $P \in \mathcal{P}_n, n>0$, $I(P) \subseteq A^n$.
- if $c \in \mathcal{F}_0$, $I(c) \in A$.
- if $p \in \mathcal{P}_0$, then $I(p) \in \{F, T\} = \{0, 1\}$.

Defn: Suppose that $\mathfrak{A} = (A, I)$ and $\mathfrak{B} = (B, J)$ are structures for $\mathcal{L}$. Then $\mathfrak{A} \subseteq \mathfrak{B}$ means that $A \subseteq B$ and the functions and predicates of $\mathfrak{A}$ are the restrictions of the corresponding functions and predicates of $\mathfrak{B}$.

Here $\mathfrak{A}$ is the substructure of $\mathfrak{B}$, and $\mathfrak{B}$ is the extension of $\mathfrak{A}$.

The definitions for elementary substructure (aka elementary submodel) and elementary extension are stronger: they require all definable properties of $\mathfrak{A}$ to be restrictions of the corresponding definable properties of $\mathfrak{B}$, instead of just the functions & predicates.

Def$^{\underline{n}}$: Let $\mathfrak{A}$ and $\mathfrak{B}$ be structures for $\mathcal{L}$ with $\mathfrak{A} \subseteq \mathfrak{B}$. If φ is a formula of $\mathcal{L}$, then $\mathfrak{A} \preccurlyeq_\varphi \mathfrak{B}$ means that

$$\mathfrak{A} \models \varphi[\sigma] \text{ if and only if } \mathfrak{B} \models \varphi[\sigma]$$

for all assignments σ for φ in A.

(If α is a term or a formula, an assignment for α in A is a function σ s.t. $V(\alpha) \subseteq \mathrm{dom}(\sigma)$ and $\mathrm{Ran}(\sigma) \subseteq A$. If α is a term, $V(\alpha)$ is set of variables in α, if α is a formula, set of free vars)

$\mathfrak{A} \preccurlyeq \mathfrak{B}$ means that $\mathfrak{A} \preccurlyeq_\varphi \mathfrak{B}$ for all formulas φ of $\mathcal{L}$. In this latter case, $\mathfrak{A}$ is an elementary substructure/submodel of $\mathfrak{B}$ and $\mathfrak{B}$ is an elementary extension of $\mathfrak{A}$.

Lemma: If $\mathfrak{A} \subseteq \mathfrak{B}$, and φ is quantifier-free, then $\mathfrak{A} \preccurlyeq_\varphi \mathfrak{B}$.

Def$^{\underline{n}}$: A set Σ of sentences of $\mathcal{L}$ is model-complete if and only if $\mathfrak{U} \preccurlyeq \mathfrak{B}$ whenever $\mathfrak{U}, \mathfrak{B} \models \Sigma$ and $\mathfrak{U} \subseteq \mathfrak{B}$.

How do we test if a substructure is elementary? We have the following useful lemma:

Lemma: (The Tarski-Vaught Criterion)

Let $\mathfrak{U}$ and $\mathfrak{B}$ be structures for $\mathcal{L}$ with $\mathfrak{U} \subseteq \mathfrak{B}$.

Then the following are equivalent:

① $\mathfrak{A} \preccurlyeq \mathfrak{B}$.

② For all existential formulas $\varphi(x_1, \ldots, x_n)$ where the x_i's are the free variables of φ (i.e. formulas of the form $\exists y \psi(x_1, \ldots, x_2, y)$), and for all $a_1, \ldots, a_n \in A$, if $\mathfrak{B} \models \varphi[a_1, \ldots, a_n]$, then there is some $b \in A$ such that

$$\mathfrak{B} \models \psi[a_1, \ldots, a_n, b].$$

The Löwenheim-Skolem-Tarski Theorems

We can generalize the Löwenheim-Skolem theorem we heard about 2 weeks ago.

Theorem: (the Downward Löwenheim-Skolem-Tarski Theorem)

Let $\mathfrak{B}$ be any structure for $\mathcal{L}$.

Fix κ such that

$$\max(|\mathcal{L}|, \aleph_0) \leq \kappa \leq |\mathfrak{B}|, \text{ and}$$

then fix $S \subseteq B$ with $|S| \leq \kappa$.

Then there is a $\mathfrak{A} \preccurlyeq \mathfrak{B}$ such that

$S \subseteq A$ and $|A| = \kappa$.

Proof

We use the above Lemma, the Tarski-Vaught criterion.

For each existential formula φ of $\mathcal{L}$, we choose a "skolem function" f_φ. If φ has n free variables

$x_1, \ldots, x_n$ then $\varphi = \varphi(x_1, \ldots, x_n)$ is of the form $\exists y \psi(x_1, \ldots, x_n, y)$

Applying AoC, let $f_\varphi : B^n \to B$ such that for any $(a_1, \ldots, a_n) \in B^n$, if $\mathcal{B} \models \varphi[a_1, \ldots, a_n]$ then $\mathcal{B} \models \psi[(a_1, \ldots, a_n), f_\varphi((a_1, \ldots, a_n))]$.

So if there is a $b \in B$ such that

$\mathcal{B} \models \psi[(a_1, \ldots, a_n), b]$,

then $f_\varphi((a_1, \ldots, a_n))$ chooses some b, but if not, $f_\varphi((a_1, \ldots, a_n))$ can be arbitrary.

Since $n = n_\varphi$ depends on φ, we have

$$f_\varphi : B^{n_\varphi} \longrightarrow B.$$

We will define $A \subseteq B$ to be Skolem-closed if and only if for each existential φ of $\mathcal{L}$, $f_\varphi(A^{n_\varphi}) \subseteq A$. A will be Skolem-closed.

If $n = n_\varphi = 0$, a function with 0 variables is constant, so if φ is a sentence, $\exists y \psi(y)$.

$B^0 = \{\emptyset\}$ and $f_\varphi(\emptyset) = b$ for some $b \in B$ depending on φ. If $\mathfrak{B} \models \varphi$, then $\mathfrak{B} \models \psi[b]$.

If A is Skolem-closed, then $b = f_\varphi(\emptyset) \in A$.

In particular, letting φ be $\exists y[y=y]$, we see that $A \neq \emptyset$.

Next, a Skolem-closed A is closed under all functions of $\mathcal{L}$. Let $g \in \mathcal{L}$ be an n-ary function symbol so that $g_{\mathfrak{B}} : B^n \to B$, then $g_{\mathfrak{B}}(A^n) \subseteq A$.

Proof: Let $\varphi(x_1, \ldots, x_n)$ be $\exists y[g(x_1, \ldots, x_n) = y]$ and note that the Skolem-function f_φ will be $g_{\mathfrak{B}}$.

Now we will define an $\mathcal{L}$-structure $\mathfrak{A} = (A, I)$. Let $g_{\mathfrak{A}} = g_{\mathfrak{B}}|_{A^n}$ whenever $g \in \mathcal{L}$ is an n-ary function symbol, and $P_{\mathfrak{A}} = P_{\mathfrak{B}} \cap A^n$ whenever $P \in \mathcal{L}$ is an n-ary predicate symbol.

In the case of functions, $g_{\mathfrak{B}}(A^n) \subseteq A$ means $g_{\mathfrak{A}}: A^n \to A$.

When $n=0$, g is constant, so $g_{\mathfrak{A}} = g_{\mathfrak{B}}$, in A since A is closed under functions of $\mathcal{L}$. If $p \in \mathcal{L}$ is a proposition letter, let $p_{\mathfrak{A}} = p_{\mathfrak{B}} = \{F, T\}$.

Given any Skolem-closed A, we have a structure $\mathfrak{A} \subseteq \mathfrak{B}$, then $\mathfrak{A} \preceq \mathfrak{B}$ by Tarski-Vaught.

Now we must construct an A with $S \subseteq A$ and $|A| = \kappa$. Assume $|S| = \kappa$.

Let Σ be the set of existential formulas of $\mathcal{L}$. If $T \subseteq B$, let $T' = \bigcup_{\varphi \in \Sigma} f_\varphi(T^{n_\varphi})$.

Then $T \subseteq T'$ since if $\varphi(x)$ is $\exists y[x=y]$ then $n_\varphi = 1$ and f_φ is the identity function.

If $|T| = \kappa$, then $|T'| = \kappa$.

So let $S = S_0 \subseteq S_1 \subseteq S_2 \subseteq \ldots$ where $S_{i+1} = (S_i)'$

for $i \in \omega$, and let $A = \bigcup_i S_i$.

Then each $|S_i| = \kappa$, so $|A| = \kappa$.

Since A is Skolem-closed, each $a_1, \ldots, a_n \in A$ lies in some $S_i^{n_\varphi}$, so

$f_\varphi((a_1, \ldots, a_n)) \in S_{i+1} \subseteq A$.

$\square$

This Result tells us that if $\mathcal{L}$ is countable and $\mathfrak{B}$ is arbitrary, then there must be a countable $\mathfrak{A}$ such that

$$\mathfrak{A} \preccurlyeq \mathfrak{B}.$$

What about the opposite direction?

Theorem: (The Upward Löwenheim-Skolem-Tarski Theorem)

Let $\mathfrak{B}$ be an infinite structure for $\mathcal{L}$.

Fix $\kappa \geq \max(|\mathcal{L}|, |\mathfrak{B}|)$.

Then there exists a structure $\mathfrak{C}$ for $\mathcal{L}$ such that

$\mathfrak{B} \preccurlyeq \mathfrak{C}$ and $|\mathfrak{C}| = \kappa$.

Proof

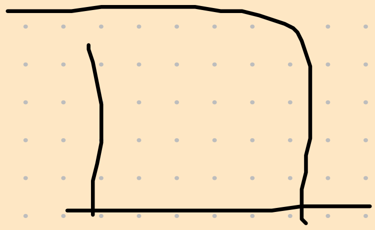
First, we need a definition:

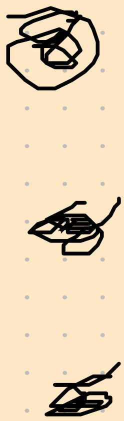
Defn: For any lexicon $\mathcal{L}$ and any structure $\mathfrak{A}$ for $\mathcal{L}$, the natural expansion of $\mathfrak{A}$ is the expansion of $\mathfrak{A}$ to $\mathcal{L}_A = \mathcal{L} \cup \{c_a : a \in A\}$ by interpreting c_a's $a \in A$, then the elementary diagram of $\mathfrak{A}$, $eDiag(\mathfrak{A})$ is $Th(\mathfrak{A}_A)$ - the set of all $\mathcal{L}_A$-sentences true in $\mathfrak{A}_A$.

Now for the proof: Let $eDiag(\mathfrak{B})$ be the elementary diagram of $\mathfrak{B}$, written in

$$\mathcal{L}_B = \mathcal{L} \cup \{c_b : b \in B\}$$

Then $eDiag(\mathfrak{B})$ has infinite model $\mathfrak{B}$ and $\kappa \geq \max(|\mathcal{L}_B|, \aleph_0)$ tells us that the standard Löwenheim-Skolem theorem implies that $\mathrm{eDiag}(\mathfrak{B})$ has a model $\mathfrak{C}$ of size κ, and $\mathfrak{C} \models \mathrm{eDiag}(\mathfrak{B})$ implies $\mathfrak{B} \preccurlyeq \mathfrak{C}$.





Models of Set Theory

Let's work a little bit with the lexicon $\mathcal{L} = \{\in\}$, the language of set theory.

If A is a nonempty set, we can view A as an $\mathcal{L}$-structure.

Def$^{\underline{n}}$: An $\in$-model is any structure $\mathfrak{U}$ for $\mathcal{L} = \{\in\}$ such that

$$\in_{\mathfrak{U}} = \{(a,b) \in A \times A : a \in b\}.$$

We write A instead of $\mathfrak{U}$, e.g.

$$A \models \varphi.$$

Def$^{\underline{n}}$: A transitive model is any $\in$-model A such that A is transitive.

$\rightsquigarrow$ Z is a transitive set if and only if $\forall xy\,[x \in y \wedge y \in z \rightarrow x \in z]$.

Consider all formulas φ of $\mathcal{L}$ in which all quantifiers are bounded, or occur in the combinations

$\forall x \in y$ or $\exists x \in y$. We call these formulas Δ_0.

Def^{n}: Let $\mathcal{L} = \{\in\}$. The Δ_0 formulas of $\mathcal{L}$ are those constructed by the following rules.

① All atomic formulas are Δ_0 formulas.

$\longrightarrow$ Recall from Gabriel's Lecture that atomic formulas are sequences of the form $p\tau_1 \ldots \tau_n$ where $n \geq 0$ and $\tau_1, \ldots, \tau_n$ are terms of $\mathcal{L}$ and either $p \in \mathcal{P}$ or p is $=$ and $n = 2$.

(2) If φ is a Δ_0 formula, and x and y are two distinct variables, $\forall x \in y \varphi$ and $\exists x \in y \varphi$ are Δ_0 formulas.

(3) If φ is Δ_0, so is $\neg\varphi$.

(4) If φ and ψ are Δ_0, so are $\varphi \wedge \psi$, $\varphi \vee \psi$, $\varphi \to \psi$, and $\varphi \leftrightarrow \psi$.

Here are some examples:

(1) $\forall z \in x (z \neq z)$

means "x is empty."

(2) $\forall z \in x (z \in y)$

means $x \subseteq y$.

(3) $\forall y \in x (\forall z \in y (z \in x))$

means "x is transitive."

④ $x \in z \wedge y \in z \wedge \forall u \in z \, (u = x \vee u = y)$

means $z = \{x, y\}$.

⑤ $\exists y \in x \, \forall z \in x \, (z = y)$

means "x is a singleton."

Lemma: If $A \subseteq B$ and A is transitive, then $A \prec_\varphi B$ for all Δ_0 formulas φ in the language of set theory.

$\rightsquigarrow$ Definitions expressible through Δ_0 formulas have the same meaning in any transitive model A.

Many set-theoretic notions can be expressed through Δ_0 formulas, including:

- x is a function
- $z = \bigcup y$
- x is a natural number
- $z = x \times y$, etc.

In last week's lecture, Cassandra mentioned the practice of viewing symbolic expressions as mathematical objects and gave the example of a polynomial ring $F[x]$ over a field F.

$\rightsquigarrow$ skipped II.14

$\rightsquigarrow$ Universal Algebra

Thank you.

— William Dudarov